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DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

UNIFORM APPROXIMATIONS OF THE STATIONARY AND PALM DISTRIBUTIONS OF MARKED POINT PROCESSES

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Gert Nieuwenhuis

FEW 543

Stationary Point
Approximation Theory

Refereed by Prof.dr. B.B. van der Genugten

Uniform Approximations of the Stationary and Palm Distributions of Marked Point Processes

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Summary

Let P be the distribution of a stationary marked point process on $\mathbf{R}$ and let P_L^0 be its Palm distribution with respect to a set L of marks. The probability measures $P_{i,L}$, $i \in \mathbf{Z}$, arise from P by shifting the origin to the ith occurrence with mark in L. In Nieuwenhuis (1991) the well-known approximation of P_L^0 by the mean of $P_{1,L}, \ldots, P_{n,L}$ under an ergodicity condition was proved to be uniform. If this condition is not satisfied, then the (uniform) limit of this mean can still be characterized.

In this paper it is proved that in the results just mentioned P may be replaced by $P_{L'}^0$, where L' is another set of marks with $L \cap L' = \emptyset$. In a 'dual' theorem the roles of P and P_L^0 are interchanged. Starting from P_L^0 the uniform convergence of a Césaro mean of shifted probabilities is considered. Under a weak ergodicity condition the limit is equal to P.

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1 Introduction

Let K be a complete and separable metric space. A *marked point process on* $\mathbf{R}$ *with mark space* K is a random element Φ in the class of all integer-valued measures φ on the σ-field $\operatorname{Bor}\mathbf{R} \times \operatorname{Bor} K$ such that:

$$\varphi(A \times K) < \infty \text{ for all bounded } A \in \operatorname{Bor}\mathbf{R}.$$

Let M_K be this class and endow it with the σ-field $\mathcal{M}_K$ generated by the sets $[\varphi(A \times L) = k] := \{\varphi \in M_K : \varphi(A \times L) = k\}$, $k \in \mathbf{N}_0$, $L \in \operatorname{Bor} K$ and $A \in \operatorname{Bor}\mathbf{R}$. The distribution of Φ will be denoted by P, a probability measure on $(M_K, \mathcal{M}_K)$.

For $\varphi \in M_K$ and $L \in \operatorname{Bor} K$ we define $\tilde{\varphi}_L \in M_K$ and the counting measure φ_L on $\operatorname{Bor}\mathbf{R}$ by $\tilde{\varphi}_L(B) := \varphi(B \cap (\mathbf{R} \times L))$ and $\varphi_L(A) := \varphi(A \times L)$, $B \in \operatorname{Bor}\mathbf{R} \times \operatorname{Bor} K$ and $A \in \operatorname{Bor}\mathbf{R}$. Set

$$\begin{aligned} M_L^\infty &:= \{\varphi \in M_K : \varphi_L(-\infty, 0) = \varphi_L(0, \infty);\ \varphi_K(\{s\}) \le 1 \text{ for all } s \in \mathbf{R}\}, \\ M_L^0 &:= \{\varphi \in M_L^\infty : \varphi_L(\{0\}) = 1\}, \\ \mathcal{M}_L^\infty &:= M_L^\infty \cap \mathcal{M}_K \text{ and } \mathcal{M}_L^0 := M_L^0 \cap \mathcal{M}_K, \end{aligned}$$

$L \in \operatorname{Bor} K$. Define $\lambda(L) := \mathbf{E}\Phi_L(0,1]$. It will always be assumed that Φ (or rather P) is *stationary* (i.e., $\Phi(t + \cdot) =_d \Phi$ for all $t \in \mathbf{R}$), that $P(M_K^\infty) = 1$, and that the *intensity* $\lambda(K)$ is finite. We will only consider $L \in \operatorname{Bor} K$ with $P(M_L^\infty) = 1$. The atoms of $\varphi \in M_K^\infty$ are denoted by $(X_i(\varphi), k_i(\varphi)) \in \mathbf{R} \times K$, $i \in \mathbf{Z}$, with the convention

$$\ldots < X_{-1}(\varphi) < X_0(\varphi) \le 0 < X_1(\varphi) < \ldots.$$

$X_i(\varphi)$ is interpreted as the ith *occurrence* (or *point*) of φ, $k_i(\varphi)$ as the accessory *mark*. For $\varphi \in M_L^\infty$ we write $X_i^L(\varphi) := X_i(\tilde{\varphi}_L)$, the '$i$th L-point of φ', and $\alpha_i^L(\varphi) := X_{i+1}^L(\varphi) - X_i^L(\varphi)$.

Two types of shifts will be considered. The *time shifts* $T_t : M_K^\infty \to M_K^\infty$, $t \in \mathbf{R}$, are determined by $T_t\varphi(A \times L) := \varphi((t + A) \times L)$, $\varphi \in M_K^\infty$, $A \in \operatorname{Bor}\mathbf{R}$ and $L \in \operatorname{Bor} K$. Note that $\varphi(t + \cdot) = T_t\varphi$ can be represented by $\{(X_i(\varphi) - t, k_i(\varphi)) : i \in \mathbf{Z}\}$. For fixed $L \in \operatorname{Bor} K$ with $P(M_L^\infty) = 1$ the *point shifts* $\vartheta_{n,L} : M_L^\infty \to M_L^\infty$, $n \in \mathbf{Z}$, are defined by

$\vartheta_{n,L}\varphi := \varphi(X_n^L(\varphi) + \cdot)$. The probability measures $P_{n,L} := P\vartheta_{n,L}^{-1}$, $n \in \mathbf{Z}$, on $(M_L^\infty, \mathcal{M}_L^\infty)$ arise from P by shifting the origin to the nth occurrence.

Let $L \in \operatorname{Bor} K$ be such that $P(M_L^\infty) = 1$. We now consider the Palm distribution P_L^0 of Φ with respect to L. The formal definition of P_L^0 follows below, but intuitively P_L^0 is the conditional distribution of Φ given the occurrence of an arbitrary L-point in the origin. This intuitive meaning of P_L^0 can (at least in the ergodic case) indeed be confirmed, since $P_L^0(B)$, $B \in \mathcal{M}_L^\infty$, can be approximated by

$$\frac{1}{n}\sum_{i=1}^{n} P[\vartheta_{n,L}\varphi \in B], \qquad n \in \mathbf{N}, \tag{1.1}$$

if Φ satisfies some weak ergodicity condition. Here $[\vartheta_{n,L}\varphi \in B] := \{\varphi \in M_L^\infty : \vartheta_{n,L}\varphi \in B\}$. In Nieuwenhuis (1991) it has been proved that the convergence in question holds uniformly for $B \in \mathcal{M}_L^\infty$. If, however, this ergodicity condition is not satisfied, then the (uniform) limit $Q_L^0(B)$ of the sequence in (1.1) is not necessarily equal to $P_L^0(B)$, but can still be characterized. See Theorem 1.2 below.

In Section 4 it is proved that in the above results (as stated in Theorem 1.2) P may be replaced by $P_{L'}^0$, where $L' \in \operatorname{Bor} K$ is another set of marks, $L \cap L' = \emptyset$. The resulting theorem concerns the limit behavior of $(n^{-1}\sum_{i=1}^{n} P_{L'}^0[\vartheta_{n,L}\varphi \in B])_{n\in\mathbf{N}}$. It can be compared with Theorem 3 in Konstantopoulos and Walrand (1988), which in essence concerns weak convergence of $(P_{L'}^0[\vartheta_{n,L}\varphi \in \cdot])_{n\in\mathbf{N}}$ under some mixing condition. See also König and Schmidt (1986).

In Section 3 the roles of P and P_L^0 in Theorem 1.2 are interchanged. It is proved that the Césaro mean

$$\frac{1}{t}\int_0^t P_L^0[T_x\varphi \in B]dx, \qquad t \in (0,\infty), \tag{1.2}$$

tends to some limit $Q_L(B)$ as $t \to \infty$, uniformly for $B \in \mathcal{M}_L^\infty$. The probability measure Q_L turns out to be equal to P if some weak ergodicity condition holds. The probability measures Q_L and Q_L^0 are connected.

Our treatment involves conditioning on invariant σ-fields. Some preliminary lemmas are proved in Section 2.

In our proves we have to go from P_L^0 to P or from P to $P_{L'}^0$ several times. The method used to bridge these gaps (the 'Radon-Nikodym approach', see Nieuwenhuis (1991; Section 1)), is a consequence of Theorem 1.1.

We next formalize some of the notions mentioned above and give some other definitions and notations.

For $L \in \mathrm{Bor}\, K$ with $P(M_L^\infty) = 1$ the *Palm distribution* P_L^0 *of* Φ (or rather P) *with respect to* L is defined by

$$P_L^0(A) := \frac{1}{\lambda(L)} \mathbf{E}\left[\sum_{i=1}^{\Phi((0,1]\times L)} 1_A(\vartheta_{i,L}\Phi)\right], \qquad A \in \mathcal{M}_L^\infty. \tag{1.3}$$

Intuitively P_L^0 arises from P by shifting the origin to an arbitrary L-point. Note the difference between P_L^0 and $P_{0,L}$, in notation as well as in interpretation. Several probability measures on $(M_L^\infty, \mathcal{M}_L^\infty)$ have been defined now: P, P_L^0, $P_{n,L}$. In this research expectations with respect to these measures are denoted by E, E_L^0, $E_{n,L}$, respectively. When another probability measure Q on $(M_L^\infty, \mathcal{M}_L^\infty)$ is considered, we will write E_Q. Expectation with respect to an universal probability space $(\Omega, \mathcal{F}, P)$ is (as in (1.3)) denoted by $\mathbf{E}$.
Note that $P_L^0(M_L^0) = 1$. The probability measure P_L^0 has the following properties:

$$P_L^0 \vartheta_{n,L}^{-1} = P_L^0 \qquad \text{for all } n \in \mathbf{Z}, \tag{1.4}$$

$$P(A) = \lambda(L) \int_0^\infty P_L^0[X_1^L(\varphi) > u;\ \varphi(u + \cdot) \in A]du, \qquad A \in \mathcal{M}_L^\infty. \tag{1.5}$$

See Franken et al. (1982), Matthes, Kerstan and Mecke (1978), Kallenberg (1983/86), and Brandt, Franken and Lisek (1990) for more information.
The *inversion formula* (1.5) expresses P in terms of P_L^0; the definition in (1.3) expresses P_L^0 in terms of P. There is another way of going from P_L^0 to P (and vice versa). The essence of the approach in question is contained in the next theorem. It is proved in Nieuwenhuis (1989), with an extension to **marked** point processes in Nieuwenhuis (1991; Section 5).
(Two probability distributions Q_1 and Q_2 on a common measurable space are said to be equivalent (notation $Q_1 \sim Q_2$) if they have the same null-sets. A Radon-Nikodym derivative of Q_1 with respect to Q_2 is denoted by $\frac{dQ_1}{dQ_2}$.)

Theorem 1.1 *Let $n \in \mathbb{Z}$ and let $L \in \operatorname{Bor} K$ be such that $P(M_L^\infty) = 1$. Then*
(i) $P_{n,L} \sim P_L^0$,
(ii) $\dfrac{dP_{n,L}}{dP_L^0} = \lambda(L)\alpha_{-n}^L \quad P_L^0$-a.s.

Suppose that $f : M_L^0 \to \mathbf{R}$ is P_L^0-integrable. We are now able to express the P_L^0-expectation of f in terms of a P-expectation:

$$E_L^0 f = \frac{1}{\lambda(L)} E_{0,L}\left(\frac{1}{\alpha_0^L} f\right) = \frac{1}{\lambda(L)} E\left(\frac{1}{\alpha_0^L} f \circ \vartheta_{0,L}\right). \tag{1.6}$$

Reversely, if $g : M_L^\infty \to \mathbf{R}$ is P-integrable with $Eg = Eg \circ \vartheta_{0,L}$, then P-expectation of g can be transformed to a P_L^0-expectation:

$$Eg = E_{0,L} g = \lambda(L) E_L^0(\alpha_0^L g). \tag{1.7}$$

For more information we refer to Nieuwenhuis (1991). The approach in (1.6) and (1.7), where $P_{0,L}$ is used as a bridge between P_L^0 and P, is very common in this research.

Consider the following invariant σ-fields:

$$\begin{aligned} \mathcal{I}_L' &:= \{A \in \mathcal{M}_L^\infty : T_t^{-1} A = A \text{ for all } t \in \mathbf{R}\} \text{ and} \\ \mathcal{I}_L &:= \{A \in \mathcal{M}_L^\infty : \vartheta_{1,L}^{-1} A = A\}. \end{aligned} \tag{1.8}$$

Φ (or rather P) is called *ergodic* if $P(A) \in \{0,1\}$ for all $A \in \mathcal{I}_K'$; it is called *pseudo- L -ergodic* if

$$E_L^0(\alpha_0^L | \mathcal{I}_L) = \frac{1}{\lambda(L)} \quad P_L^0 - \text{a.s.} \tag{1.9}$$

P_L^0 is called *ergodic* if $P_L^0(A) \in \{0,1\}$ for all $A \in \mathcal{I}_L$.

The following theorem has been the inspiration and motivation for this research. In this *cross-convergence* result P_L^0 is approximated when starting from P. It is proved in Nieuwenhuis (1991; Sections 4 and 5).

Theorem 1.2 *Let $L \in \operatorname{Bor} K$ be such that $P(M_L^\infty) = 1$. Then*

$$\frac{1}{n}\sum_{i=1}^{n} P[\vartheta_{i,L}\varphi \in B] \to E(E_L^0(1_B|\mathcal{I}_L)) =: Q_L^0(B) \tag{1.10}$$

uniformly for $B \in \mathcal{M}_L^\infty$. The probability measure Q_L^0 on $(M_L^\infty, \mathcal{M}_L^\infty)$ is equivalent to P_L^0 and

$$\frac{dQ_L^0}{dP_L^0} = \lambda(L)E_L^0(\alpha_0^L|\mathcal{I}_L) \quad P_L^0 - a.s.$$

Q_L^0 and P_L^0 are equal iff Φ is pseudo- L -ergodic.

Let Q_1 and Q_2 be probability measures on a common measurable space, both dominated by a σ-finite measure μ and having densities h_1 and h_2 respectively. The *total variation distance* between Q_1 and Q_2 is defined by

$$d(Q_1, Q_2) := \int |h_1 - h_2| d\mu.$$

It is well-known that

$$d(Q_1, Q_2) = 2\sup_A |Q_1(A) - Q_2(A)| = 2(Q_1[h_1 \geq h_2] - Q_2[h_1 \geq h_2]). \tag{1.11}$$

As a final remark we note that, when talking about Radon-Nikodyn derivatives, the supplement a.s. (almost surely) is often suppressed.

2 Conditioning on invariant σ-fields

One of the objectives of this research is to obtain some uniform cross- convergence results without assuming ergodicity. To realize this in this general setting we will condition on invariant σ-fields.

Recall the definitions of $\mathcal{I}_L$ and $\mathcal{I}'_L$ in (1.8).

Lemma 2.1 *Let $L \in \operatorname{Bor} K$. Then:*
(a) If $A \in \mathcal{I}_L$, then $\vartheta_{i,L}^{-1} A = A$ for all $i \in \mathbf{Z}$.
(b) $\mathcal{I}_L = \mathcal{I}'_L$.

Proof. Let $A \in \mathcal{I}_L$. For $\psi \in M_L^\infty$ we have

$$\psi \in A \quad \text{iff} \quad \vartheta_{1,L}\psi \in A. \tag{2.1}$$

Let $\varphi \in M_L^\infty$. Since $\vartheta_{1,L} = \vartheta_{1,L} \circ \vartheta_{0,L}$, we obtain (by applying (2.1) twice) that

$$\varphi \in A \qquad \text{iff} \qquad \vartheta_{1,L}(\vartheta_{0,L}\varphi) \in A \qquad \text{iff} \qquad \vartheta_{0,L}\varphi \in A.$$

Consequently, $\vartheta_{0,L}^{-1} A = A$.
Since $\vartheta_{0,L} = \vartheta_{1,L} \circ \vartheta_{-1,L}$, we have for $\varphi \in M_L^\infty$, as another consequence of (2.1),

$$\varphi \in A \qquad \text{iff} \qquad \vartheta_{1,L}(\vartheta_{-1,L}\varphi) \in A \qquad \text{iff} \qquad \vartheta_{-1,L}\varphi \in A.$$

Hence, $\vartheta_{-1,L}^{-1} A = A$. Since $\vartheta_{i,L} = \vartheta_{1,L}^i$ and $\vartheta_{-i,L} = \vartheta_{-1,L}^i$ for all $i \geq 1$, part (a) follows immediately.

For $A \in \mathcal{I}'_L$ we have: $\varphi \in A$ iff $T_t\varphi \in A$, for all $t \in \mathbf{R}$ and $\varphi \in M_L^\infty$. Consequently, $\varphi \in A$ iff $\vartheta_{1,L}\varphi \in A$, for all $\varphi \in M_L^\infty$. So, $\vartheta_{1,L}^{-1} A = A$ and $\mathcal{I}'_L \subset \mathcal{I}_L$.
Let $A \in \mathcal{I}_L$. By (a) we have

$$\varphi \in A \qquad \text{iff} \qquad \vartheta_{i,L}\varphi \in A, \qquad \text{for all } \varphi \in M_L^\infty \text{ and } i \in \mathbf{Z}. \tag{2.2}$$

Let $\varphi \in M_L^\infty$ and $t \in \mathbf{R}$. Take $i \in \mathbf{Z}$ such that $X_{i-1}^L(\varphi) \leq t < X_i^L(\varphi)$. Then

$$T_t\varphi \in A \qquad \text{iff} \qquad \vartheta_{1,L}(T_t\varphi) \in A \qquad \text{iff} \qquad \vartheta_{i,L}\varphi \in A \qquad \text{iff} \qquad \varphi \in A,$$

cf. (2.2). So, $T_t^{-1} A = A$ and $\mathcal{I}_L \subset \mathcal{I}'_L$. Part (b) follows immediately. □

Note that as a consequence of Lemma 2.1 every $\mathcal{I}_L$-measurable function $f : M_L^\infty \to [0,\infty)$ satisfies

$$f \circ \vartheta_{i,L}(\varphi) = f(\varphi) \qquad \text{and} \qquad f \circ T_t(\varphi) = f(\varphi) \tag{2.3}$$

for all $\varphi \in M_L^\infty$, $i \in \mathbf{Z}$, and $t \in \mathbf{R}$.

In view of Section 4 we next consider two disjoint sets of marks. So, let L, $L' \in \operatorname{Bor} K$ and $L \cap L' = \emptyset$. Furthermore, set

$$\begin{aligned} M_{L,L'}^\infty &:= M_L^\infty \cap M_{L'}^\infty \text{ and } \mathcal{M}_{L,L'}^\infty := M_{L,L'}^\infty \cap \mathcal{M}_K, \\ \mathcal{I}_{L,L'} &:= \{A \in \mathcal{M}_{L,L'}^\infty : \vartheta_{1,L}^{-1} A = A\}, \\ \mathcal{I}'_{L,L'} &:= \{A \in \mathcal{M}_{L,L'}^\infty : T_t^{-1} A = A \text{ for all } t \in \mathbf{R}\}. \end{aligned}$$

In this context the maps $\vartheta_{1,L}$ and T_t, $t \in \mathbf{R}$, will always be restricted to $M_{L,L'}^\infty$. The following relations can easily be proved:

$$\begin{array}{ll} \mathcal{I}'_L \cap M_{L'}^\infty = \mathcal{I}'_{L,L'} & \text{and } \ \mathcal{I}_L \cap M_{L'}^\infty = \mathcal{I}_{L,L'}; \\ \mathcal{I}'_{L,L'} \subset \mathcal{I}'_L & \text{and } \ \mathcal{I}_{L,L'} \subset \mathcal{I}_L \end{array} \tag{2.4}$$

At first sight part (b) of the next lemma seems rather surprising.

Lemma 2.2 *Let* $L, L' \in \operatorname{Bor} K$ *with* $L \cap L' = \emptyset$. *Then:*
(a) If $A \in \mathcal{I}_{L,L'}$, then $\vartheta_{i,L}^{-1} A = A$ for all $i \in \mathbf{Z}$;
(b) $\mathcal{I}'_{L,L'} = \mathcal{I}_{L,L'} = \mathcal{I}_{L',L}$.

Proof. The proof of (a) is similar to the proof of Lemma 2.1(a). Part (b) is an immediate consequence of Lemma 2.1(b) and (2.4) since

$$\begin{aligned} \mathcal{I}_{L,L'} &= \mathcal{I}_L \cap M_{L'}^\infty = \mathcal{I}'_L \cap M_{L'}^\infty = \mathcal{I}'_{L,L'} = \mathcal{I}'_{L',L} \\ &= \mathcal{I}'_{L'} \cap M_L^\infty = \mathcal{I}_{L'} \cap M_L^\infty = \mathcal{I}_{L',L}. \end{aligned}$$

□

Next a stationary point process Φ with distribution P is put upon the stage. Suppose that $P(M_L^\infty) = 1$. Since $\mathcal{I}'_L \subset \mathcal{I}'_K$ and $\mathcal{I}'_L = \mathcal{I}'_K \cap \mathcal{M}_L^\infty$, the σ-field $\mathcal{I}'_K$ in the definition of ergodicity of P in Section 1 may equivalently be replaced by $\mathcal{I}'_L$. As a consequence of Lemma 2.1(b) we obtain:

$$\begin{aligned} P &\text{ is ergodic} &\iff& P_L^0 \text{ is ergodic,} \\ P &\text{ is ergodic} &\implies& P \text{ is pseudo-}L\text{-ergodic.} \end{aligned} \tag{2.5}$$

See also Nieuwenhuis (1991; Section 4).

In the following lemma some special conditional expectations are compared. The random variable $N_1^L : M_L^\infty \to \mathbf{N}_0$ is defined by $N_1^L(\varphi) := \varphi_L(0,1]$.

Lemma 2.3 *Let L, $L' \in \operatorname{Bor} K$ with $L \cap L' = \emptyset$ and $P(M_{L,L'}^\infty) = 1$. Then:*

(a) $E_L^0(\alpha_0^L|\mathcal{I}_L) > 0$ and $E(\frac{1}{\alpha_0^L}|\mathcal{I}_L) = E(N_1^L|\mathcal{I}_L) = \frac{1}{E_L^0(\alpha_0^L|\mathcal{I}_L)}$ $\qquad$ P- and P_L^0-a.s.

(b) $E(\frac{1}{\alpha_0^L}|\mathcal{I}_{L,L'}) = \frac{1}{E_L^0(\alpha_0^L|\mathcal{I}_{L,L'})}$ $\qquad$ $P_{L'}^0$-a.s.

Proof. Let $A \in \mathcal{I}_L$. Note that $\alpha_0^L = \alpha_0^L \circ \vartheta_{0,L}$. By (2.3), Theorem 1.1, and (1.3) we have

$$E(1_A E(\frac{1}{\alpha_0^L}|\mathcal{I}_L)) = E(1_A \frac{1}{\alpha_0^L}) = E_{0,L}(1_A \frac{1}{\alpha_0^L}) = \lambda(L) P_L^0(A) = E(1_A N_1^L).$$

Under P the first equality in the right-hand part of (a) is a consequence of this observation. Since P_L^0 and $P_{0,L}$ have the same null-sets, this equality is also valid under P_L^0. (The left-hand part of (2.3) is used here.) Set $B := [E_L^0(\alpha_0^L|\mathcal{I}_L) \le 0]$. Then

$$0 \ge E_L^0(1_B E_L^0(\alpha_0^L|\mathcal{I}_L)) = E_L^0(1_B \alpha_0^L).$$

Since $P_L^0[\alpha_0^L > 0] = 1$, we obtain

$$P_L^0(B^c) = 1 \qquad \text{and} \qquad P(B^c) = E(1_{B^c} \circ \vartheta_{0,L}) = P_{0,L}(B^c) = 1.$$

The left-hand part of (a) follows.

Let again $A \in \mathcal{I}_L$. By (2.3) and Theorem 1.1 we have

$$
\begin{aligned}
E\left(1_A \frac{1}{E_L^0(\alpha_0^L|\mathcal{I}_L)}\right) &= E\left(1_A \circ \vartheta_{0,L} \frac{1}{E_L^0(\alpha_0^L|\mathcal{I}_L) \circ \vartheta_{0,L}}\right) \\
&= \lambda(L) E_L^0 \left(\alpha_0^L 1_A \frac{1}{E_L^0(\alpha_0^L|\mathcal{I}_L)}\right) \\
&= \lambda(L) P_L^0(A) = E\left(1_A \frac{1}{\alpha_0^L}\right).
\end{aligned}
$$

In the third equality we conditioned on $\mathcal{I}_L$. The right-hand part of (a) follows immediately.

Next (b). Since $\mathcal{I}_{L,L'} = \mathcal{I}_L \cap M_{L'}^{\infty}$ and $P(M_{L,L'}^{\infty}) = 1$, it is obvious that (a) remains valid if $\mathcal{I}_L$ is replaced by $\mathcal{I}_{L,L'}$. So, part (b) holds under P. Since $\mathcal{I}_{L,L'} = \mathcal{I}_{L',L} \subset \mathcal{I}_{L'}$, both conditional expectations in (b) are $\mathcal{I}_{L'}$-measurable. Hence, equality holds $P_{0,L'}$-a.s. as well. By Theorem 1.1(ii) part (b) follows. □

The following equality is proved in Nieuwenhuis (1991).

$$
E_L^0\left(\int_0^{\alpha_0^L} g \circ T_s ds | \mathcal{I}_L\right) = E(g|\mathcal{I}_L) E_L^0(\alpha_0^L|\mathcal{I}_L) \qquad P\text{- and } P_L^0\text{- a.s.} \tag{2.6}
$$

Here $g : M_L^{\infty} \to \mathbf{R}$ is P-integrable. It holds as well with $\mathcal{I}_L$ replaced by $\mathcal{I}_{L,L'}$. According to the above reference Relation (2.6) can be considered as a conditional version of the inversion formula (1.5).

3 Approximation of P

In this section convergence of the Cesaro mean $t^{-1} \int_0^t P_L^0[T_x\varphi \in B]dx$ will be considered. The limit $Q_L(B)$ is related to the limit $Q_L^0(B)$ of (1.1).

By a generalization of Theorem 3.1 in Nieuwenhuis (1991) (see Section 5 of this reference) it is obvious that

$$
\frac{1}{t}\int_0^t 1_B \circ T_x dx \longrightarrow E(1_B|\mathcal{I}_L) \qquad P\text{- and } P_L^0\text{-a.s.} \tag{3.1}
$$

for all $B \in \mathcal{M}_L^{\infty}$. The limit equals $P(B)$ if Φ is ergodic. By dominated convergence we have

$$
\frac{1}{t}\int_0^t P_L^0[T_x\varphi \in B]dx \longrightarrow E_L^0[E(1_B|\mathcal{I}_L)] =: Q_L(B) \tag{3.2}
$$

for all $B \in \mathcal{M}_L^\infty$. Q_L is a probability measure on $(M_L^\infty, \mathcal{M}_L^\infty)$. By Theorem 1.1(ii) and conditioning on $\mathcal{I}_L$ we obtain

$$Q_L(B) = \frac{1}{\lambda(L)} E\left[\frac{1}{\alpha_0^L} E(1_B | \mathcal{I}_L)\right] = \frac{1}{\lambda(L)} E\left[1_B E\left(\frac{1}{\alpha_0^L} | \mathcal{I}_L\right)\right].$$

Since $E(1/\alpha_0^L | \mathcal{I}_L) > 0$ P-a.s. (see Lemma 2.3(a)),

$$Q_L \sim P \qquad \text{and} \qquad \frac{dQ_L}{dP} = \frac{1}{\lambda(L)} E\left(\frac{1}{\alpha_0^L} | \mathcal{I}_L\right) \qquad P\text{-a.s.} \tag{3.3}$$

This observation will be used in the proof of the following theorem.

Theorem 3.1 *The convergence in* (3.2) *holds uniformly for $B \in \mathcal{M}_L^\infty$. If Φ is pseudo-L-ergodic, then $Q_L = P$.*

Proof. Since

$$\frac{1}{t} \int_0^t P_L^0[T_x \varphi \in B] dx = \frac{1}{\lambda(L)t} \int_0^t E\left(\frac{1}{\alpha_0^L} 1_B \circ T_x \circ \vartheta_{0,L}\right) dx,$$

it is sufficient to prove that (3.4) and (3.5) below are satisfied:

$$\sup_{B \in \mathcal{M}_L^\infty} \frac{1}{\lambda(L)t} \left| \int_0^t E\left(\frac{1}{\alpha_0^L} 1_B \circ T_x \circ \vartheta_{0,L}\right) dx - \int_0^t E\left(\frac{1}{\alpha_0^L} 1_B \circ T_x\right) dx \right| \to 0, \tag{3.4}$$

$$\sup_{B \in \mathcal{M}_L^\infty} \left| \frac{1}{\lambda(L)t} \int_0^t E\left(\frac{1}{\alpha_0^L} 1_B \circ T_x\right) dx - Q_L(B) \right| \to 0, \tag{3.5}$$

as $t \to \infty$. First note that

$$\int_0^t 1_B \circ T_x \circ \vartheta_{0,L} dx = \int_{X_0^L}^{X_0^L + t} 1_B \circ T_x dx.$$

Hence,

$$\frac{1}{\lambda(L)t\alpha_0^L}\left|\int_0^t 1_B \circ T_x \circ \vartheta_{0,L}dx - \int_0^t 1_B \circ T_x dx\right| \leq$$
$$\leq \frac{2}{\lambda(L)\alpha_0^L}1_{[|X_0^L|>t]} + \frac{2|X_0^L|}{\lambda(L)t\alpha_0^L}1_{[|X_0^L|\leq t]}$$
$$\leq \frac{2}{\lambda(L)\alpha_0^L}1_{[|X_0^L|>t]} + \frac{2}{\lambda(L)t}$$

for all $B \in \mathcal{M}_L^\infty$. As a consequence the left-hand side of (3.4) is bounded from above by

$$\frac{2}{\lambda(L)}E\left(\frac{1}{\alpha_0^L}1_{[\alpha_0^L>t]}\right) + \frac{2}{\lambda(L)t} = 2P_L^0[\alpha_0^L > t] + \frac{2}{\lambda(L)t},$$

which tends to 0 as $t \to \infty$.
Next (3.5). Set

$$\hat{Q}_{t,L}(B) := \frac{1}{\lambda(L)t}\int_0^t E\left(\frac{1}{\alpha_0^L}1_B \circ T_x\right)dx.$$

Note that $\hat{Q}_{t,L}$ is a probability measure on $(M_L^\infty, \mathcal{M}_L^\infty)$. By (3.1) and dominated convergence we obtain

$$\hat{Q}_{t,L}(B) \longrightarrow E\left[\frac{1}{\lambda(L)\alpha_0^L}E(1_B|\mathcal{I}_L)\right] \text{ as } t \to \infty. \tag{3.6}$$

Since $E(1_B|\mathcal{I}_L) = E(1_B|\mathcal{I}_L) \circ \vartheta_{0,L}$, the limit in (3.6) equals $Q_L(B)$. By stationarity and Fubini's theorem we have

$$\hat{Q}_{t,L}(B) = \frac{1}{\lambda(L)t}\int_0^t E\left(\frac{1}{\alpha_0^L \circ T_{-x}}1_B\right)dx = E\left(1_B\frac{1}{\lambda(L)t}\int_0^t \frac{1}{\alpha_0^L \circ T_{-x}}dx\right).$$

So, $\hat{Q}_{t,L}$ is dominated by P and

$$\frac{d\hat{Q}_{t,L}}{dP} = \frac{1}{\lambda(L)t}\int_0^t \frac{1}{\alpha_0^L \circ T_{-x}}dx \qquad P\text{-a.s.} \tag{3.7}$$

By an ergodic results similar to (3.1) we obtain

$$\frac{1}{t}\int_0^t \frac{1}{\alpha_0^L \circ T_{-x}}dx \longrightarrow E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right) \qquad P\text{-a.s.}$$

Observe that

$$E\left(\frac{1}{n}\int_0^n \frac{1}{\alpha_0^L \circ T_{-x}}dx\right) = \frac{1}{n}\int_0^n E\left(\frac{1}{\alpha_0^L}\right)dx = \lambda(L) = E\left(E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right)\right)$$

for all $n \in \mathbb{N}$. Lemma A2.1 in Brandt, Franken and Lisek (1990) ensures that $\left(n^{-1}\int_0^n(\alpha_0^L \circ T_{-x})^{-1}dx\right)_{n\in\mathbb{N}}$ is uniformly P-integrable. By (1.11), (3.7) and (3.3) it is obvious that

$$d(\hat{Q}_{t,L}, Q_L) = E\left|\frac{1}{\lambda(L)t}\int_0^t \frac{1}{\alpha_0^L \circ T_{-x}}dx - \frac{1}{\lambda(L)}E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right)\right| \to 0$$

as $\mathbb{N} \ni t \to \infty$, which proves (3.5) for discrete time-parameter. The transition to continuous time-parameter follows immediately.

The second part of the theorem is a consequence of (3.3), the definition of pseudo-L-ergodicity in (1.9), and Lemma 2.3(a). □

Note that by stationarity of P and the right-hand part of (2.3),

$$\begin{aligned} Q_L[T_a\varphi \in B] &= \frac{1}{\lambda(L)}E\left[E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right)1_B \circ T_a\right] \\ &= \frac{1}{\lambda(L)}E\left[E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right)1_B\right] = Q_L(B) \end{aligned}$$

for all $B \in \mathcal{M}_L^\infty$. Hence, Q_L is stationary. Since $Q_L = P$ and $Q_L^0 = P_L^0$ (see (1.10)) provided that Φ is pseudo-L-ergodic, one might wonder if Q_L^0 is the Palm distribution with respect to L associated with Q_L.

To prove that this is usually not the case, let $\tilde{Q}_L^0$ be this Palm distribution associated with Q_L and let $\tilde{\lambda}(L)$ be the intensity of Q_L. Recall the definition of N_1^L in Section 2. By (3.3), conditioning on $\mathcal{I}_L$, Theorem 1.1, and Lemma 2.3 we have

$$\tilde{\lambda}(L) = E_{Q_L}N_1^L = \frac{1}{\lambda(L)}E\left(\frac{1}{\alpha_0^L}E(N_1^L|\mathcal{I}_L)\right) = E_L^0(E(N_1^L|\mathcal{I}_L)) = E_L^0(1/E_L^0(\alpha_0^L|\mathcal{I}_L)).$$

Since (cf. (1.3) and (3.3))

$$\begin{aligned}
\tilde{Q}_L^0(B) &= \frac{1}{\tilde{\lambda}(L)} E_{Q_L} \sum_{i=1}^{N_1^L} 1_B \circ \vartheta_{i,L} \\
&= \frac{1}{\tilde{\lambda}(L)\lambda(L)} E\left[\sum_{i=1}^{N_1^L} 1_B \circ \vartheta_{i,L} \cdot E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right)\right] \\
&= \frac{1}{\tilde{\lambda}(L)\lambda(L)} E\left[\sum_{i=1}^{N_1^L} \left(1_B E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right)\right) \circ \vartheta_{i,L}\right] \\
&= \frac{1}{\tilde{\lambda}(L)} E_L^0\left(1_B E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right)\right)
\end{aligned}$$

for all $B \in \mathcal{M}_L^\infty$, we obtain

$$\tilde{Q}_L^0(B) = \frac{E_L^0(1_B / E_L^0(\alpha_0^L|\mathcal{I}_L))}{E_L^0(1/E_L^0(\alpha_0^L|\mathcal{I}_L))}, \qquad B \in \mathcal{M}_L^\infty. \tag{3.8}$$

Consequently,

$$\tilde{Q}_L^0 \sim P_L^0 \qquad \text{and} \qquad \frac{d\tilde{Q}_L^0}{dP_L^0} = \frac{1/E_L^0(\alpha_0^L|\mathcal{I}_L)}{E_L^0(1/E_L^0(\alpha_0^L|\mathcal{I}_L))}. \tag{3.9}$$

Hence (cf. Theorem 1.2),

$$\frac{d\tilde{Q}_L^0}{dQ_L^0} = \frac{d\tilde{Q}_L^0}{dP_L^0}\frac{dP_L^0}{dQ_L^0} = \frac{1/(E_L^0(\alpha_0^L|\mathcal{I}_L))^2}{\lambda(L)E_L^0(1/E_L^0(\alpha_0^L|\mathcal{I}_L))}$$

and

$$\tilde{Q}_L^0 = Q_L^0 \qquad \text{iff} \qquad \Phi \text{ is pseudo-}L\text{-ergodic.}$$

This last result ensures that Q_L^0 is the Palm distribution with respect to L associated with Q_L iff Φ is pseudo-L-ergodic.

4 Approximation of P_L^0 starting from $P_{L'}^0$

Let $L, L' \in \text{Bor}\, K$ be such that $L \cap L' = \emptyset$ and $P(M_{L,L'}^\infty) = 1$. In this section our emphasis is on the Palm distributions P_L^0 and $P_{L'}^0$, now considered as probability measures on $(M_{L,L'}^\infty, \mathcal{M}_{L,L'}^\infty)$. Relation (1.10) expresses uniform approximation of P_L^0, starting from P. It holds as well with $P_{L'}^0$ instead of P.

Since the restriction of $E_L^0(1_B|\mathcal{I}_L)$ to $M_{L,L'}^\infty$ equals $E_L^0(1_B|\mathcal{I}_{L,L'})$ P_L^0- and P-a.s., the following cross ergodic theorem holds:

$$\frac{1}{n}\sum_{i=1}^{n} 1_B \circ \vartheta_{i,L} \longrightarrow E_L^0(1_B|\mathcal{I}_{L,L'}) \quad P\text{-a.s.}, \tag{4.1}$$

$B \in \mathcal{M}_{L,L'}^\infty$. Our starting point is now $P_{L'}^0$. So, we are interested in the probability measure $Q_{L,L'}$ defined by

$$Q_{L,L'}(B) := E_{L'}^0[E_L^0(1_B|\mathcal{I}_{L,L'})], \quad B \in \mathcal{M}_{L,L'}^\infty. \tag{4.2}$$

Since $\mathcal{I}_{L,L'} = \mathcal{I}_{L',L}$ (cf. Lemma 2.2), it follows that

$$P_{0,L'}(A) = E(1_A \circ \vartheta_{0,L'}) = E(1_A) = E(1_A \circ \vartheta_{0,L}) = P_{0,L}(A)$$

for all $A \in \mathcal{I}_{L,L'}$. Set $M^0 := M_L^0 \cap M_{L'}^\infty$ and $A_0 := [E_L^0(1_{M^0}|\mathcal{I}_{L,L'}) = 1]$. Since $P_L^0(A_0) = 1$, $A_0 \in \mathcal{I}_{L,L'}$, and $P_L^0 \sim P_{0,L}$, we obtain: $P_{L'}^0(A_0) = 1$. Consequently, $Q_{L,L'}(M^0) = 1$. Using the Radon-Nikodym derivatives $dP_{L'}^0/dP_{0,L'}$ and $dP_{0,L}/dP_L^0$, cf. Theorem 1.1, we obtain

$$\begin{aligned}
Q_{L,L'}(B) &= \frac{1}{\lambda(L')} E\left[\frac{1}{\alpha_0^{L'}} E_L^0(1_B|\mathcal{I}_{L,L'})\right] \\
&= \frac{1}{\lambda(L')} E\left[E\left(\frac{1}{\alpha_0^{L'}}|\mathcal{I}_{L,L'}\right) E_L^0(1_B|\mathcal{I}_{L,L'})\right] \\
&= \frac{\lambda(L)}{\lambda(L')} E_L^0\left[\alpha_0^L E\left(\frac{1}{\alpha_0^{L'}}|\mathcal{I}_{L,L'}\right) E_L^0(1_B|\mathcal{I}_{L,L'})\right] \\
&= \frac{\lambda(L)}{\lambda(L')} E_L^0\left[E_L^0(1_B E_L^0(\alpha_0^L|\mathcal{I}_{L,L'}) E\left(\frac{1}{\alpha_0^{L'}}|\mathcal{I}_{L,L'}\right) |\mathcal{I}_{L,L'})\right]
\end{aligned}$$

$$= \frac{\lambda(L)}{\lambda(L')} E_L^0 \left[1_B \frac{E_L^0(\alpha_0^L | \mathcal{I}_{L,L'})}{E_{L'}^0(\alpha_0^{L'} | \mathcal{I}_{L,L'})} \right],$$

$B \in \mathcal{M}_{L,L'}^\infty$. In the second and fourth equality we conditioned on $\mathcal{I}_{L,L'}$; the last equality is a consequence of Lemma 2.3(b). Consequently, on $(M_{L,L'}^\infty, \mathcal{M}_{L,L'}^\infty)$ we have:

$$Q_{L,L'} \sim P_L^0 \qquad \text{and} \qquad \frac{dQ_{L,L'}}{dP_L^0} = \frac{\lambda(L) E_L^0(\alpha_0^L | \mathcal{I}_{L,L'})}{\lambda(L') E_{L'}^0(\alpha_0^{L'} | \mathcal{I}_{L,L'})}. \tag{4.3}$$

This observation is essential for the following theorem.

Theorem 4.1
(a) *Relation* (4.1) *holds as well with* $P_{L'}^0$ *instead of* P.
(b) *The convergence*

$$\frac{1}{n} \sum_{i=1}^{n} P_{L'}^0[\vartheta_{i,L}\varphi \in B] \longrightarrow Q_{L,L'}(B) \tag{4.4}$$

is uniform for $B \in \mathcal{M}_{L,L'}^\infty$.
(c) *If* Φ *is pseudo-L-ergodic and pseudo-L'-ergodic, then* $Q_{L,L'}$ *equals* P_L^0.

Proof. Let $\nu : M_{L,L'}^\infty \to \mathbf{N}_0$ be defined by $\nu(\varphi) := \varphi_L(X_0^{L'}(\varphi), 0]$, $\varphi \in M_{L,L'}^\infty$. Since $\vartheta_{i,L} \circ \vartheta_{0,L'}\varphi = \varphi(X_{i-\nu(\varphi)}^L(\varphi) + \cdot)$, we obtain

$$P_{0,L'} \left[\tfrac{1}{n} \sum_{i=1}^{n} 1_B \circ \vartheta_{i,L} \to E_L^0(1_B | \mathcal{I}_{L,L'}) \right]$$

$$= P \left[\tfrac{1}{n} \sum_{i=1}^{n} 1_B \circ \vartheta_{i-\nu,L} \to E_L^0(1_B | \mathcal{I}_{L,L'}) \right]$$

$$= P \left[\tfrac{1}{n} \sum_{j=1-\nu}^{n-\nu} 1_B \circ \vartheta_{j,L} \to E_L^0(1_B | \mathcal{I}_{L,L'}) \right],$$

$B \in \mathcal{M}_{L,L'}^\infty$. The last probability equals 1 since (4.1) remains valid if in the summation n is replaced by $n - \nu$. Since $P_{L'}^0 \sim P_{0,L'}$, part (a) follows immediately.

By dominated convergence it is obvious that (4.4) holds for all $B \in \mathcal{M}^{\infty}_{L,L'}$; only the uniformity of the convergence remains to prove. Define

$$R_n(B,\varphi) := \frac{1}{\lambda(L')\alpha_0^{L'}(\varphi)}\frac{1}{n}\sum_{j=1-\nu(\varphi)}^{n-\nu(\varphi)} 1_B \circ \vartheta_{j,L}(\varphi),$$

$$S_n(B,\varphi) := \frac{1}{\lambda(L')\alpha_0^{L'}(\varphi)}\frac{1}{n}\sum_{j=1}^{n} 1_B \circ \vartheta_{j,L}(\varphi),$$

for $\varphi \in M^{\infty}_{L,L'}$ and $B \in \mathcal{M}^{\infty}_{L,L'}$. Note that the left-hand side in (4.4) equals $ER_n(B)$. Since

$$|R_n(B,\varphi) - S_n(B,\varphi)| \leq \frac{2}{\lambda(L')\alpha_0^{L'}(\varphi)}1_{[\nu\geq n]}(\varphi) + \frac{2\nu(\varphi)}{n\lambda(L')\alpha_0^{L'}(\varphi)}1_{[\nu<n]}(\varphi)$$

for all $\varphi \in M^{\infty}_{L,L'}$, we obtain:

$$E|R_n(B) - S_n(B)| \leq \frac{2}{\lambda(L')}E\left[\frac{1}{\alpha_0^{L'}}1_{[\nu\geq n]}\right] + \frac{2}{n\lambda(L')}E\left[\frac{\nu}{\alpha_0^{L'}}\right],$$

uniformly for $B \in \mathcal{M}^{\infty}_{L,L'}$. This upper bound tends to 0 as $n \to \infty$. Hence, it is sufficient to prove that

$$\sup_{B\in\mathcal{M}^{\infty}_{L,L'}} |ES_n(B) - Q_{L,L'}(B)| \to 0 \quad \text{as } n \to \infty . \tag{4.5}$$

For this we define probability measures $\mu_{i,L}$ as follows

$$\mu_{i,L}(B) := E\left[\frac{1}{\lambda(L')\alpha_0^{L'}}1_B \circ \vartheta_{i,L}\right], \qquad B \in \mathcal{M}^{\infty}_{L,L'}.$$

By (1.5), Fubini's theorem, and (1.4) we obtain:

$$\begin{aligned}
\mu_{i,L}(B) &= \frac{\lambda(L)}{\lambda(L')}\int_0^{\infty} E_L^0\left[\frac{1}{\alpha_0^{L'}\circ T_u}(1_B \circ \vartheta_{i,L}\circ T_u)1_{[\alpha_0^L>u]}\right]du \\
&= \frac{\lambda(L)}{\lambda(L')}E_L^0\left[\int_0^{\alpha_0^L}\frac{1}{\alpha_0^{L'}\circ T_u}1_B \circ \vartheta_{i,L}du\right] \\
&= \frac{\lambda(L)}{\lambda(L')}E_L^0(1_B\eta_i),
\end{aligned}$$

where

$$\eta_i = \int_0^{\alpha_{-i}^L} \frac{1}{\alpha_0^{L'} \circ T_u \circ \vartheta_{-i,L}} du.$$

Hence, for all i and $n \in \mathbf{N}$ we have

$$\mu_{i,L} \ll P_L^0 \qquad \text{and} \qquad \frac{d\mu_{i,L}}{dP_L^0} = \frac{\lambda(L)}{\lambda(L')}\eta_i, \tag{4.6}$$

$$ES_n \ll P_L^0 \qquad \text{and} \qquad \frac{dES_n}{dP_L^0} = \frac{\lambda(L)}{\lambda(L')}\frac{1}{n}\sum_{i=1}^{n}\eta_i. \tag{4.7}$$

By (4.3), (4.7), and the definition of d just above (1.11) we obtain

$$d(ES_n, Q_{L,L'}) = \frac{\lambda(L)}{\lambda(L')} E_L^0 \left| \frac{1}{n}\sum_{i=1}^{n}\eta_i - \frac{E_L^0(\alpha_0^L|\mathcal{I}_{L,L'})}{E_{L'}^0(\alpha_0^{L'}|\mathcal{I}_{L,L'})} \right|. \tag{4.8}$$

Note that (η_i) is P_L^0-stationary. So,

$$\frac{1}{n}\sum_{i=1}^{n}\eta_i \longrightarrow E_L^0\left(\int_0^{\alpha_0^L} \frac{1}{\alpha_0^{L'} \circ T_u} du|\mathcal{I}_{L,L'}\right) \qquad P_L^0\text{- a.s.} \tag{4.9}$$

By (2.6) and Lemma 2.3(b) the limit in (4.9) is equal to

$$E\left(\frac{1}{\alpha_0^{L'}}|\mathcal{I}_{L,L'}\right) E_L^0(\alpha_0^L|\mathcal{I}_{L,L'}) = \frac{E_L^0(\alpha_0^L|\mathcal{I}_{L,L'})}{E_{L'}^0(\alpha_0^{L'}|\mathcal{I}_{L,L'})} \qquad P_L^0\text{- a.s.} \tag{4.10}$$

Since the sequence $(\sum_{i=1}^{n}\eta_i/n)$ is uniformly P_L^0- integrable (cf., e.g., Brandt, Franken and Lisek (1990; L.A2.1)), we obtain by (4.8)-(4.10) that

$$d(ES_n, Q_{L,L'}) \to 0 \qquad \text{as } n \to \infty,$$

see also Brémaud (1981; Th.T26). This proves (4.5) and hence part (b).
Part (c) is an immediate consequence of (4.3) and Lemma 2.3. □

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